QUANTUM COMPUTING

Student Workbook

Every question in the course, with no answer and no solution. Work each one on the page, then check it against the artifact or against the instructor edition.

Contains	Level	Answers
120 questions across 6 modules	Undergraduate	Not printed in this edition

1 The Mathematics of Quantum States
20 questions

2 States, Measurement and Dynamics
20 questions

3 Mixed States and Entanglement
20 questions

4 The Bloch Sphere and Quantum Gates
20 questions

5 Circuits and Protocols
20 questions

6 Quantum Algorithms
20 questions

This course is an adaptation. Its syllabus and the sequence of topics it teaches derive from **Quantum Computing Lectures** by Aleksandr Krasnok, github.com/AlexKrasnok/quantum-computing-lectures, used under CC BY 4.0. Every scene, figure, laboratory and question here is written for this artifact.

How to use it

The questions are in the order the course meets them, and each is labelled with what it asks for. Only the statement and its lettered parts are printed; the reasoning stays for you to supply. The question numbers are shared with every other edition, so D5-04 is the same question in the artifact, in this workbook and in the instructor solutions.

MODULE 1

The Mathematics of Quantum States

20 questions on the mathematics of quantum states. Write your reasoning in the space under each one.

D1–01 · INNER PRODUCTS, NORMS AND ORTHOGONALITY

Two states of one qubit are

$$|a\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}, \qquad |b\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}.$$

a) Compute $\langle a|b\rangle$.

b) Compute $\langle b|a\rangle$, and state the relation between the two.

c) Give $|\langle a|b\rangle|^2$ and say what it means about telling the two states apart.

D1–02 · INNER PRODUCTS, NORMS AND ORTHOGONALITY

An unnormalised column is $|\tilde{\psi}\rangle = 3|0\rangle + 4i|1\rangle$.

a) Give $\langle\tilde{\psi}|\tilde{\psi}\rangle$ and the length $\|\tilde{\psi}\|$.

b) Give the normalised state $|\psi\rangle$.

c) Give the two computational-basis probabilities that Chapter 2 will read off it.

D1–03 · INNER PRODUCTS, NORMS AND ORTHOGONALITY

A student computes the squared length of $|a\rangle = (1,\, i)$ in NumPy with `np.dot(a, a)` and gets zero.

a) Compute $\langle a|a\rangle$ correctly.

b) Compute the quantity `np.dot(a, a)` returns, and say what it would claim about $|a\rangle$.

c) Give the normalised state.

D1-04 · EXPANDING A STATE IN A BASIS

A state is $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$.

a) Confirm that it is normalised.

b) Expand it in the basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

c) Give $|\langle +|\psi\rangle|^2$ and $|\langle -|\psi\rangle|^2$, and check them.

D1-05 · EXPANDING A STATE IN A BASIS

The resolution of the identity in the X basis is $|+\rangle\langle +| + |-\rangle\langle -| = I$.

a) Insert it into $\langle 0|1\rangle$ and write the resulting sum.

b) Evaluate every factor and give the total.

c) Say what the calculation demonstrates.

d) For a complete function basis with coefficients c_n, state Parseval identity and the squared error after retaining only $n \leq N$.

D1-06 · GLOBAL AGAINST RELATIVE PHASE

Three states are

$$|\psi_1\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{i|0\rangle - |1\rangle}{\sqrt{2}}, \quad |\psi_3\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}.$$

a) Decide whether $|\psi_1\rangle$ and $|\psi_2\rangle$ are the same physical state.

b) Compute $\langle \psi_1|\psi_3\rangle$.

c) Say which pair a measurement can separate with certainty, and which it cannot separate at all.

D1-07 · GLOBAL AGAINST RELATIVE PHASE

A qubit is prepared in $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)$.

a) Give $P(0)$ and $P(1)$ in the computational basis.

b) Apply a Hadamard and give the two probabilities after it, as a function of φ.

c) Evaluate both at $\varphi = \pi/3$.

D1-08 · BUILDING AN OPERATOR AND TESTING IT

A state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$.

a) Form the operator $P = |\psi\rangle\langle\psi|$ as a 2×2 matrix.

b) Verify that $P^\dagger = P$ and $P^2 = P$.

c) Compute $P|0\rangle$ and give its length.

D1-09 · BUILDING AN OPERATOR AND TESTING IT

Two matrices are

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

a) Decide whether each is Hermitian.

b) Compute AX and XA.

c) Decide whether AX is Hermitian, and say what settles it.

D1-10 · BUILDING AN OPERATOR AND TESTING IT

A matrix is $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.

a) Write $U^\dagger$.

b) Compute $U^\dagger U$ and decide whether U is unitary.

c) Apply U to $|0\rangle$ and confirm that the length is preserved.

D1-11 · BUILDING AN OPERATOR AND TESTING IT

A matrix is $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Give $\det M$.

b) Compute $M^\dagger M$.

c) Apply M to $|1\rangle$ and give the length of the result. Is M a legal gate?

D1-12 · EIGENVALUES, THE SPECTRAL FORM, AND FUNCTIONS

A Hermitian matrix is $A = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$.

a) Give the characteristic equation and solve it.

b) Find a normalised eigenvector for each eigenvalue.

c) Check the two eigenvalues against the trace and the determinant.

D1-13 · EIGENVALUES, THE SPECTRAL FORM, AND FUNCTIONS

Use the eigenvectors of $A = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$, namely $|v_+\rangle = \frac{1}{\sqrt{2}}(1, i)$ with $\lambda = 2$ and $|v_-\rangle = \frac{1}{\sqrt{2}}(1, -i)$ with $\lambda = 0$.

a) Build the projectors P_+ and P_-.

b) Verify $P_+ + P_- = I$ and $P_+ P_- = 0$.

c) Reassemble A from the spectral form and confirm it.

D1-14 · EIGENVALUES, THE SPECTRAL FORM, AND FUNCTIONS

Continue with $A = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} = 2P_+ + 0\,P_-$.

a) Give A^2 from the spectral form, and confirm it by squaring the matrix.

b) Give $\sqrt{A}$, meaning the positive operator whose square is A.

c) Give e^{-iAt}.

D1-15 · EIGENVALUES, THE SPECTRAL FORM, AND FUNCTIONS

For $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, compute e^X.

a) Give the eigenvalues and the two projectors of X.

b) Give e^X from the spectral form, in closed form and to four decimals.

c) Give the matrix that exponentiating X entry by entry would produce, and show by its eigenvalues that it cannot be the exponential of any Hermitian matrix.

D1-16 · TENSOR PRODUCTS, BIT ORDER AND COUNTING

A two-qubit register has q_1 in $|1\rangle$ and q_0 in $|0\rangle$.

a) Write the joint state as a four-entry column, in the ordering this course fixes.

b) Say which entry carries the amplitude, and why.

c) Give the column the opposite convention would produce, and name the state it describes.

D1-17 · TENSOR PRODUCTS, BIT ORDER AND COUNTING

Consider

$$|\alpha\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle-|1\rangle}{\sqrt{2}}, \qquad |\beta\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}.$$

a) Write $|\alpha\rangle$ as a four-entry column.

b) Show that $|\beta\rangle$ cannot be written as a product of two single-qubit states.

c) Arrange the four amplitudes of each state as a 2×2 array $c_{q_1 q_0}$ and give its determinant.

D1-18 · TENSOR PRODUCTS, BIT ORDER AND COUNTING

A register holds n qubits, and a simulator stores each complex amplitude as two double-precision numbers.

a) Give the number of amplitudes for $n = 1, 2, 3, 4$.

b) Give the number of amplitudes and the storage in bytes for $n = 20$.

c) Give the number of bits one measurement of the register returns, and say what that implies for an algorithm.

D1-19 · A FULL-LENGTH QUESTION COMBINING SEVERAL OF THE SHAPES

A qubit is prepared in

$$|\psi\rangle = \cos\tfrac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}\,|1\rangle, \qquad \theta = \tfrac{\pi}{3}, \quad \varphi = \tfrac{\pi}{2}.$$

a) Give the state as an explicit column.

b) Give $P(0)$ and $P(1)$.

c) Give $P(+)$ and $P(-)$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$.

d) The state is now multiplied by $e^{i\pi/4}$. State which of the four probabilities change.

D1-20 · A FULL-LENGTH QUESTION COMBINING SEVERAL OF THE SHAPES

An operator is

$$G = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

a) Decide whether G is Hermitian.

b) Compute G^2, and use it to give the eigenvalues of G without solving the characteristic equation.

c) Give e^{-iGt} in closed form.

d) Evaluate it at $t = \pi/(2\sqrt{2})$ and confirm that the result is unitary.

MODULE 2

States, Measurement and Dynamics

20 questions on states, measurement and dynamics. Write your reasoning in the space under each one.

D2-01 · BORN PROBABILITIES IN A NAMED BASIS

A qubit is prepared in $|\psi\rangle = \frac{1}{5}\,(3|0\rangle + 4i|1\rangle)$.

a) Give the probabilities of the two outcomes of a Z measurement.

b) Give the probabilities of the two outcomes of an X measurement.

c) Is the state an eigenstate of either? Say how you can tell without solving anything.

D2-02 · BORN PROBABILITIES IN A NAMED BASIS

A qubit in $|0\rangle$ is measured along the direction $\mathbf{n} = (\sin\alpha,\ 0,\ \cos\alpha)$.

a) Write the two projectors of that measurement.

b) Give $p(+1)$ as a function of α, and evaluate it at $\alpha = 60^\circ$.

c) At which α is the measurement a fair coin?

D2-03 · BORN PROBABILITIES IN A NAMED BASIS

A qubit is prepared in $|{+i}\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + i|1\rangle)$.

a) Give the outcome probabilities for a Z measurement.

b) Give them for an X measurement.

c) Give them for a Y measurement, and say what the three answers together mean.

D2-04 · THE STATE AFTER A MEASUREMENT, AND SEQUENCES

A qubit is prepared in $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ and measured in the X basis.

a) Give the two outcome probabilities.

b) Give the state after each outcome.

c) The outcome was $+1$. A Z measurement now follows. Give its probabilities.

D2-05 · THE STATE AFTER A MEASUREMENT, AND SEQUENCES

A qubit is prepared in $|0\rangle$. Three measurements are made in a row: Z, then X, then Z again.

a) Give the outcome of the first measurement and its probability.

b) Give the probabilities of the second, and the state after each.

c) Give the probability that the third measurement agrees with the first.

d) Give the same probability when the middle measurement is omitted.

D2-06 · THE STATE AFTER A MEASUREMENT, AND SEQUENCES

A readout reports the wrong bit with probability $\epsilon = 0.05$, in either direction, so its effects are $E_0 = (1 - \epsilon)|0\rangle\langle 0| + \epsilon|1\rangle\langle 1|$ and $E_1 = I - E_0$.

a) Give the probability of reporting 0 for a qubit truly in $|0\rangle$.

b) For a state with true $p(0) = q$, give the reported probability.

c) A long run reports 0 a fraction 0.62 of the time. Give the corrected estimate of q, and say what the correction does to the error bar.

d) Do the effects determine the state after outcome m? Name the extra object and give its update rule.

D2-07 · EXPECTATION VALUES AND VARIANCES

A qubit is in $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$.

a) Give $\langle Z\rangle$.

b) Give $\langle X\rangle$ and $\langle Y\rangle$.

c) Give the Bloch vector and check it.

D2-08 · EXPECTATION VALUES AND VARIANCES

Continue with $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$, for which $\langle Z\rangle = \frac{1}{2}$, $\langle X\rangle = \frac{\sqrt{3}}{2}$ and $\langle Y\rangle = 0$.

a) Give $\mathrm{Var}(Z)$ and ΔZ.

b) Give $\mathrm{Var}(X)$ and ΔX.

c) Give $\Delta X\,\Delta Z$ and the Robertson bound, and say what the comparison shows.

D2-09 · EXPECTATION VALUES AND VARIANCES

An observable and a state are

$$H = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}, \qquad |\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

a) Give $\langle H\rangle$.

b) Give $\langle H^2\rangle$ and $\mathrm{Var}(H)$.

c) Give the eigenvalues of H, and say what they show about the answer to (a).

D2-10 · COMMUTATORS, COMPATIBILITY AND THE BOUND

Use the Pauli product rule $\sigma_i\sigma_j = \delta_{ij}I + i\varepsilon_{ijk}\sigma_k$.

a) Give $[X, Y]$, $[Y, Z]$ and $[Z, X]$.

b) Give the anticommutator $\{X, Y\}$.

c) Do X and Y share an eigenvector? Say how the answer follows from (a).

D2-11 · COMMUTATORS, COMPATIBILITY AND THE BOUND

Two pairs of observables are $A = Z$ with $B = 3Z + 2I$, and $A = Z$ with $B = X$.

a) Give $[Z,\, 3Z + 2I]$.

b) Give the common eigenvectors of the first pair, with the eigenvalue of each observable.

c) Give $[Z, X]$, and say whether any state makes both readings certain.

D2-12 · COMMUTATORS, COMPATIBILITY AND THE BOUND

A qubit is in $|{+i}\rangle = \frac{1}{\sqrt{2}}\,(|0\rangle + i|1\rangle)$.

a) Give $\langle X\rangle$, $\langle Y\rangle$ and $\langle Z\rangle$.

b) Give ΔX and ΔZ.

c) Compare $\Delta X\,\Delta Z$ with $\frac{1}{2}\,|\langle [X, Z]\rangle|$.

D2-13 · EVOLUTION, STATIONARY STATES AND PULSES

A qubit evolves under $H = \frac{\omega}{2} Z$ from the initial state $|+\rangle$.

a) Give $U(t)$.

b) Give $|\psi(t)\rangle$, with the global phase removed.

c) Give $P(0)$ and $P(+)$ as functions of t.

d) Give the first time at which the state is $|-\rangle$.

D2-14 · EVOLUTION, STATIONARY STATES AND PULSES

A system has $H|E_1\rangle = E_1|E_1\rangle$ and $H|E_2\rangle = E_2|E_2\rangle$, with $E_2 > E_1$.

a) Show that $|E_1\rangle$ is stationary.

b) Give $|\psi(t)\rangle$ for the initial state $\frac{1}{\sqrt{2}}\left(|E_1\rangle + |E_2\rangle\right)$, with the global phase removed.

c) Give the period of any observable quantity, and say what happens to it when both energies are raised by the same amount.

d) For an infinite square well of width a, give $\phi_n(x)$ and E_n, and show that $E_2 = 4E_1$.

e) In the coordinate representation, give $\hat{p}$ and the free-particle Hamiltonian. Why is one plane wave not a normalisable physical state on the whole line?

D2-15 · EVOLUTION, STATIONARY STATES AND PULSES

A qubit starting in $|0\rangle$ is driven on resonance by $H = \frac{\Omega}{2}X$.

a) Give $U(t)$.

b) Give $P(1)$ as a function of t.

c) Give the first time at which the qubit has been flipped, and the operator U is then equal to.

d) Give the first time at which the state is an even superposition.

D2-16 · EVOLUTION, STATIONARY STATES AND PULSES

The same qubit is driven off resonance, by $H = \frac{1}{2}\left(\Omega_x X + \Delta Z\right)$ with $\Omega_x = 1$ and $\Delta = 1$, starting from $|0\rangle$.

a) Give the generalised rate Ω and the rotation axis $\mathbf{n}$.

b) Give the largest value $P(1)$ ever reaches.

c) Give the first time at which it reaches that value.

d) Say what a π pulse calibrated on resonance does here.

D2-17 · WHAT A FINITE RUN IS WORTH

A circuit whose true outcome probability is near $\frac{1}{2}$ is run for $N = 10\,000$ shots.

a) Give the standard error of the estimated probability.

b) Give the number of shots needed to reach a standard error of 0.001.

c) The readout has a known bias of 0.02. Say what running more shots does to it.

D2-18 · WHAT A FINITE RUN IS WORTH

The expectation value of a Pauli observable is estimated from N shots by averaging the ± 1 readings. The true value is $\langle X \rangle = 0.8$.

a) Give $p(+1)$.

b) Give the standard error of the estimate for $N = 1000$.

c) Give the number of shots needed for a standard error of 0.01.

D2-19 · A FULL-LENGTH QUESTION: ONE EXPERIMENT END TO END

A qubit is prepared in $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$ with $\theta = \frac{\pi}{2}$ and $\varphi = \frac{\pi}{3}$.

a) Give the Bloch vector.

b) Give the outcome probabilities of an X measurement.

c) Give $\langle X \rangle$.

d) Give the number of shots needed to estimate $\langle X \rangle$ to a standard error of 0.02.

D2-20 · A FULL-LENGTH QUESTION: ONE EXPERIMENT END TO END

A qubit starting in $|0\rangle$ is driven on resonance by $H = \frac{\Omega}{2}X$ with $\Omega = 2$, held for a time t, and then measured in the computational basis.

a) Give $|\psi(t)\rangle$.

b) Give $P(1)$ and $\langle Z \rangle$ as functions of t.

c) Give the first time at which $P(1) = \frac{1}{2}$.

d) At that time, give the number of shots needed to estimate $\langle Z \rangle$ to a standard error of 0.05.

MODULE 3

Mixed States and Entanglement

20 questions on mixed states and entanglement. Write your reasoning in the space under each one.

D3-01 · BUILDING A DENSITY OPERATOR, AND TESTING ONE

A source emits $|0\rangle$ with probability $\frac{1}{4}$ and $|+\rangle$ with probability $\frac{3}{4}$, and nothing records which.

a) Write the density operator as a matrix in the computational basis.

b) Is the state pure? Say how you can tell without diagonalising.

c) Give the purity, and check it a second way.

D3-02 · BUILDING A DENSITY OPERATOR, AND TESTING ONE

Three candidate matrices: $M_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}$, $M_2 = \begin{bmatrix} 0.5 & 0.6i \\ -0.6i & 0.5 \end{bmatrix}$, $M_3 = \begin{bmatrix} 0.5 & 0.3 \\ 0.3 & 0.6 \end{bmatrix}$.

a) Say which of the three are density operators, and which condition each failure breaks.

b) For the one that is a state, give its eigenvalues.

c) Give its purity, and check it against the eigenvalues.

D3-03 · BUILDING A DENSITY OPERATOR, AND TESTING ONE

Device A emits $|+\rangle$ or $|-\rangle$, each with probability $\frac{1}{2}$. Device B emits $|0\rangle$ or $|1\rangle$, each with probability $\frac{1}{2}$.

a) Write both density operators.

b) Is there any measurement, or any number of copies, that tells the two devices apart?

c) Device A is now retuned to emit $|+\rangle$ with probability $\frac{3}{4}$. Give the new matrix, and name a measurement that now separates it from device B.

D3-04 · PREDICTIONS FROM A DENSITY OPERATOR

A qubit is in the state $\rho = \begin{bmatrix} 0.7 & 0.2i \\ -0.2i & 0.3 \end{bmatrix}$.

a) Give the three Pauli expectation values.

b) Give the outcome probabilities of a Z measurement.

c) Give the purity, and say whether the state is pure.

D3-05 · PREDICTIONS FROM A DENSITY OPERATOR

A qubit is in the mixture $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$, and is measured along the direction $\mathbf{n} = (\sin\alpha,\ 0,\ \cos\alpha)$.

a) Give the Bloch vector of the state.

b) Give $p(+1)$ as a function of α, and evaluate it at $\alpha = 60^\circ$.

c) At which α is the measurement a fair coin, and why does the answer not depend on how mixed the state is?

D3-06 · PREDICTIONS FROM A DENSITY OPERATOR

The observable $A = X + Z$ is measured on the state $\rho = \begin{bmatrix} 0.625 & 0.375 \\ 0.375 & 0.375 \end{bmatrix}$.

a) Give $\langle A \rangle$.

b) Give the variance of A in this state.

c) The mean comes out as 1. Is 1 a reading the instrument can return?

D3-07 · CHANNELS, DAMPING AND DEPHASING

A qubit prepared in $|+\rangle$ passes through an amplitude-damping channel of strength γ.

a) Give the density matrix after the channel for $\gamma = 0.36$.

b) Give its purity.

c) As γ runs from 0 to 1, what is the smallest purity the output ever has, and at which γ?

D3-08 · CHANNELS, DAMPING AND DEPHASING

A qubit prepared in $|+\rangle$ passes through the phase-flip channel $\mathcal{E}_Z(\rho) = (1-p)\rho + pZ\rho Z$.

a) Give the density matrix and the purity for $p = 0.25$.

b) At which p is the output the maximally mixed state?

c) Give the output at $p = 1$, and explain why a channel at full strength has destroyed nothing.

D3-09 · CHANNELS, DAMPING AND DEPHASING

A device has a relaxation time $T_1 = 80\ \mu$s and a pure-dephasing time $T_\phi = 40\ \mu$s.

a) Give T_2.

b) A qubit is prepared with an excited population and a coherence, and idles for $20\ \mu$s. Give the surviving fraction of each.

c) What is the largest T_2 this T_1 allows, and what would have to be true of the device to reach it?

D3-10 · REDUCED STATES AND THE PARTIAL TRACE

Two qubits are in the pure state $|\psi\rangle = \frac{1}{2}\left(\sqrt{3}\,|00\rangle + |11\rangle\right)$.

a) Give the reduced state of each qubit.

b) Give the purity of each.

c) Is the pair entangled? Give the reason in one sentence.

D3-11 · REDUCED STATES AND THE PARTIAL TRACE

Two qubits are in the state $|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$.

a) Give the reduced state of the first qubit.

b) Decide whether the state is entangled, using the amplitude test.

c) Give the Schmidt coefficients and the entanglement entropy in bits.

D3–12 · REDUCED STATES AND THE PARTIAL TRACE

Compare $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ with the mixed state $\rho_{\text{mix}} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$.

a) Give the reduced state of the first qubit in each case.

b) Give the purity of each joint state.

c) Name one measurement whose statistics differ, and give both answers.

D3–13 · SEPARABILITY, SCHMIDT RANK AND ENTROPY

Three two-qubit pure states: (i) $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, (ii) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, (iii) $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$.

a) Apply the amplitude test to each.

b) Factor the ones that factor.

c) For each, say what the reduced state of the first qubit is without computing a partial trace.

D3–14 · SEPARABILITY, SCHMIDT RANK AND ENTROPY

Two qubits are in the pure state $|\psi\rangle = \frac{1}{\sqrt{6}}(|00\rangle + |01\rangle + 2|10\rangle)$.

a) Write the coefficient matrix and give ρ_A.

b) Give the two Schmidt coefficients.

c) Give the entanglement entropy in bits.

D3–15 · SEPARABILITY, SCHMIDT RANK AND ENTROPY

A family of two-qubit states, $|\psi(\theta)\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle$, with θ between 0 and 90°.

a) Give the Schmidt coefficients as functions of θ.

b) At which θ is the pair maximally entangled, and at which is it a product?

c) Give the entropy at $\theta = 30^\circ$.

D3-16 · BELL CORRELATIONS AND THE CHSH NUMBER

Two qubits are in the singlet $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

a) Give $\langle X \otimes X\rangle$, $\langle Y \otimes Y\rangle$ and $\langle Z \otimes Z\rangle$.

b) Give the reduced state of each qubit.

c) What do the three answers in (a) say about measuring both qubits along the *same* arbitrary direction?

D3-17 · BELL CORRELATIONS AND THE CHSH NUMBER

Both parties measure $|\Phi^+\rangle$ along directions in the z–x plane, at angles from z of $A_0 = 0°$, $A_1 = 90°$, $B_0 = 30°$ and $B_1 = -30°$. For this state the correlation of two such readings is the cosine of the angle between the directions.

a) Give the four correlations.

b) Give $S = \langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle$, and say whether it violates the classical bound.

c) Keeping A_0 and A_1 fixed and the two B angles symmetric at $\pm\varphi$, find the φ that makes S largest, and the value there.

D3-18 · BELL CORRELATIONS AND THE CHSH NUMBER

Two parties each hold one qubit of $|\Phi^+\rangle$. The second party measures their qubit in the X basis.

a) Rewrite $|\Phi^+\rangle$ with both qubits in the X basis.

b) Given that the second party obtained $+1$ and told the first party so, what is the first party's state?

c) If the second party says nothing, what is the first party's state — and does the answer depend on whether the second party measured X, measured Z, or did nothing at all?

D3-19 · A FULL-LENGTH QUESTION: ONE PAIR, END TO END

A Bell pair $|\Phi^+\rangle$ is made. One of the two qubits then passes through a phase-flip channel of strength p before both are measured along directions in the z–x plane. The channel multiplies $\langle X \otimes X\rangle$ and $\langle Y \otimes Y\rangle$ by $\eta = 1 - 2p$ and leaves $\langle Z \otimes Z\rangle$ alone.

a) With $A_0 = 0°$, $A_1 = 90°$, $B_0 = 45°$ and $B_1 = -45°$, write S as a function of η.

b) Evaluate S at $p = 0.1$.

c) Find the largest p for which the pair still violates the classical bound.

d) The state is a mixture of two Bell states and stays entangled for every $p < \frac{1}{2}$. What does the gap between that and your answer to (c) say?

D3-20 · A FULL-LENGTH QUESTION: ONE PAIR, END TO END

A qubit is prepared in $|+\rangle$ on a device with $T_1 = 50\ \mu$s and $T_2 = 30\ \mu$s. It is left idle for a time t and then measured in the X basis.

a) Give $p(+)$ as a function of t.

b) Evaluate it at $t = 20\ \mu$s.

c) How many shots are needed to estimate that probability to ± 0.01 at two standard errors?

d) Check that the two quoted times are consistent with each other, and say what would have to be measured to know T_ϕ.

MODULE 4

The Bloch Sphere and Quantum Gates

20 questions on the bloch sphere and quantum gates. Write your reasoning in the space under each one.

D4-01 · BETWEEN A STATE, A MATRIX AND A POINT

A qubit is prepared in $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}e^{i2\pi/3}|1\rangle$.

a) Give the two angles θ and φ.

b) Give the Bloch vector.

c) Give the probability of reading 0 in the computational basis, two different ways.

D4-02 · BETWEEN A STATE, A MATRIX AND A POINT

Tomography on a qubit returns $\mathbf{r} = (0.6,\ 0,\ 0.8)$.

a) Is the state pure? Say how you can tell in one line.

b) Give θ and φ, and write the state.

c) Give the probability of the $+1$ outcome of an X measurement and of a Z measurement.

D4-03 · BETWEEN A STATE, A MATRIX AND A POINT

Three pure states: $|a\rangle$ at $(\theta, \varphi) = (60^\circ, 0)$, $|b\rangle$ at $(120^\circ, 180^\circ)$, and $|c\rangle$ at $(90^\circ, 0)$.

a) Give the three Bloch vectors.

b) Which pair is orthogonal, and how do you know from the picture alone?

c) Give $|\langle c|a\rangle|^2$.

D4-04 · A GATE AS A ROTATION

The gate $R_y(70^\circ)$ is applied to $|0\rangle$.

a) Write the matrix of $R_y(\alpha)$.

b) Give the state and the Bloch vector afterwards.

c) Give $p(0)$, two ways.

D4-05 · A GATE AS A ROTATION

Consider the gate $G = \dfrac{X+Y}{\sqrt{2}}$.

a) Show that G is a rotation, and give its axis and its angle.

b) Give $G|0\rangle$.

c) Give $G|+i\rangle$, using the geometry rather than the matrices.

D4-06 · A GATE AS A ROTATION

A qubit in $|+\rangle$ has the gate T applied to it.

a) Give the Bloch vector afterwards.

b) Give the probability of reading 0 in the computational basis.

c) A Hadamard is now applied as well. Give the probability of reading 0, and say what changed.

D4-07 · COMPOSING A SEQUENCE OF GATES

The gates H, then S, then H are applied to $|0\rangle$, in that order.

a) Write the matrix product that represents the circuit.

b) Follow the Bloch vector through the three gates.

c) Name the state at the end, and say what single rotation the whole circuit is.

D4-08 · COMPOSING A SEQUENCE OF GATES

Two gates, H and S, and one input $|0\rangle$.

a) Give the state when H runs first, and when S runs first.

b) How far apart are the two results on the sphere, and what is the overlap?

c) Say in one sentence why the two orders disagree.

D4-09 · COMPOSING A SEQUENCE OF GATES

The four gates X, Z, X, Z are applied in that order.

a) Give the net unitary of the sequence.

b) What does the sequence do to any Bloch vector?

c) A control qubit is added, so that the whole four-gate sequence runs only when the control is $|1\rangle$. The control starts in $|+\rangle$. What is the control at the end?

D4-10 · DECOMPOSING AND SYNTHESISING A GATE

The S gate, $S = \mathrm{diag}(1, i)$.

a) Write S as a phase times a rotation about $\hat{z}$.

b) Give its Euler form $e^{i\alpha} R_z(\phi) R_y(\theta) R_z(\lambda)$.

c) Is the controlled-S the same circuit as the controlled-$R_z(\pi/2)$? If not, what is the difference?

D4-11 · DECOMPOSING AND SYNTHESISING A GATE

A gate $U(\theta, \phi, \lambda)$ is wanted that prepares the state at $(\theta, \varphi) = (60^\circ, 45^\circ)$ from $|0\rangle$.

a) Give the parameters θ and ϕ.

b) Is λ determined by the requirement? Say why.

c) With $\lambda = 0$, write the four entries of the matrix.

D4-12 · DECOMPOSING AND SYNTHESISING A GATE

A machine offers only H, S, T and CNOT.

a) How many T gates make a Z? Show the chain.

b) Is $\{H, S, \mathrm{CNOT}\}$ on its own universal? Say why or why not.

c) A one-qubit gate is to be approximated to ε, and the gate count grows like $\log^2(1/\varepsilon)$. By what factor does the count grow when ε drops from 10^{-2} to 10^{-4}?

D4-13 · TWO-QUBIT GATES, AND WHICH QUBIT IS WHICH

The two-qubit state is $|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$, written $|q_1 q_0\rangle$.

a) Give $(I \otimes X)|\psi\rangle$.

b) Give $(X \otimes I)|\psi\rangle$.

c) Name one measurement outcome whose probability tells the two apart.

D4-14 · TWO-QUBIT GATES, AND WHICH QUBIT IS WHICH

The state $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ is handed to a CNOT.

a) Give the result when the control is q_0 and the target is q_1.

b) Give the result when the control is q_1 and the target is q_0.

c) Which of the two outputs is entangled? Give its entanglement entropy in bits.

D4-15 · TWO-QUBIT GATES, AND WHICH QUBIT IS WHICH

The controlled-Z gate, $\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1)$, is applied to $|+\rangle \otimes |+\rangle$.

a) Give the output state.

b) Is it entangled? Give the reduced state of one qubit and the entropy.

c) CZ moves no probability at all. Reconcile that with the answer to (b).

D4-16 · TWO-QUBIT GATES, AND WHICH QUBIT IS WHICH

A chip couples q_0 to q_1 and q_1 to q_2, and nothing else. A CNOT is wanted with q_0 as control and q_2 as target.

a) Give the SWAP gate as a matrix, in the ordering $|q_1 q_0\rangle$.

b) How many CNOTs does the whole job cost if the qubits may be left in their new positions?

c) And if they must be returned to where they started?

d) A reversible half adder must also write $s = a \oplus b$ and $c = ab$. Give an embedding that preserves its inputs, and name the gates that write the two outputs.

D4-17 · ENTANGLING POWER AND UNIVERSALITY

From $|00\rangle$, the gate $R_y(\theta)$ is applied to q_0 and then a CNOT with q_0 as control and q_1 as target.

a) Give the output state in terms of θ.

b) Give the entanglement entropy at $\theta = 120^\circ$.

c) At which θ is the output a product state?

d) A further one-qubit gate is applied to each qubit. What happens to the entropy?

D4-18 · ENTANGLING POWER AND UNIVERSALITY

Four proposed gate sets: (i) every one-qubit gate; (ii) $\{H, S, \mathrm{CNOT}\}$; (iii) $\{H, S, T, \mathrm{CNOT}\}$; (iv) every one-qubit gate together with CNOT.

a) Which of the four can produce an entangled state at all?

b) Which are universal, and in which sense — exactly or to any accuracy?

c) Give one sentence saying what "universal" does not promise.

D4-19 · A FULL-LENGTH QUESTION: ONE CIRCUIT, END TO END

A single qubit starts in $|0\rangle$. The circuit applies H, then T, then H, and the qubit is measured in the computational basis.

a) Give the Bloch vector after each of the three gates.

b) Give $p(0)$ exactly, and as a decimal.

c) How many shots are needed to estimate that probability to ± 0.01 at two standard errors?

d) The T is replaced by $T^\dagger$. What happens to $p(0)$, and why?

D4-20 · A FULL-LENGTH QUESTION: ONE CIRCUIT, END TO END

Two qubits start in $|00\rangle$, written $|q_1 q_0\rangle$. The circuit applies H to q_0, then a CNOT with q_0 as control and q_1 as target, then Z to q_1.

a) Give the state after each of the three gates, and name the final one.

b) Give the two reduced states, and the entanglement in bits.

c) Give the probabilities of the four computational-basis outcomes.

d) A colleague claims the circuit produced $|\Phi^+\rangle$. Name one measurement setting that decides the question, and say how many shots it needs in the ideal case.

MODULE 5

Circuits and Protocols

20 questions on circuits and protocols. Write your reasoning in the space under each one.

D5-01 · READING AND COUNTING A CIRCUIT

A GHZ state on eight qubits is prepared two ways. The **chain** applies H to q_0 and then $\mathrm{CNOT}_{0\to1}$, $\mathrm{CNOT}_{1\to2}$, and so on up to $\mathrm{CNOT}_{6\to7}$. The **tree** applies H to q_0, then $\mathrm{CNOT}_{0\to1}$, then $\mathrm{CNOT}_{0\to2}$ and $\mathrm{CNOT}_{1\to3}$ together, then four more together.

a) Give the total gate count and the two-qubit count for each.

b) Give the depth of each.

c) A two-qubit gate takes $400\,\mathrm{ns}$ and a one-qubit gate $40\,\mathrm{ns}$. Give the duration of each.

D5-02 · READING AND COUNTING A CIRCUIT

A three-qubit circuit maps q_k to classical bit c_k, and one run prints the string 011.

a) Give the values read on each wire, and name the wire drawn at the top.

b) Give the ket and the state-vector index this string names.

c) Write the matrix of $\mathrm{CNOT}_{0\to1}$ on two qubits in the ordered basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

D5-03 · READING AND COUNTING A CIRCUIT

From $|00\rangle$, a circuit applies H to q_0, then $\mathrm{CNOT}_{0\to1}$, then Z to q_1.

a) Give the state after each of the three gates.

b) Name the final state.

c) Give the four measurement probabilities, and the depth of the circuit.

D5-04 · SHOTS, ERROR BARS AND SIMULATION COST

A circuit produces an outcome with true probability $p = 0.2$.

a) Give the standard error of the estimated probability after 1000 shots.

b) Give the shots needed for a standard error of 0.002.

c) The measured value comes out at 0.31 after $40{,}000$ shots. Say what that does and does not mean.

D5-05 · SHOTS, ERROR BARS AND SIMULATION COST

An exact simulation stores each complex amplitude in 16 bytes.

a) Give the memory a 34-qubit state vector needs.

b) Give the largest n whose state vector fits in $1\,\mathrm{TB} = 10^{12}$ bytes.

c) Give the memory the circuit matrix of a 15-qubit circuit needs, and say why circuits are checked on states instead.

D5-06 · SHOTS, ERROR BARS AND SIMULATION COST

A one-qubit dynamic circuit applies H to a qubit in $|0\rangle$, measures it into c_0, applies X if $c_0 = 1$, and measures again into c_1.

a) Give the joint distribution of (c_1, c_0).

b) Give it again with the conditional X removed.

c) Give it again with the conditional X kept and one further H inserted before the second measurement.

D5-07 · REWRITING AND ROUTING FOR A MACHINE

A circuit has 12 CNOTs and 20 one-qubit gates. The machine offers only $R_z(\lambda)$, $R_x(\pi/2)$ and CZ. Each CNOT becomes $H\,\mathrm{CZ}\,H$, each H becomes three rotations, and each of the original one-qubit gates becomes three rotations.

- **a)** Give the instruction count after translation, and the two-qubit count.
- **b)** A two-qubit gate fails with probability 0.008 and a one-qubit gate with 0.0003. Give the probability that no gate fails.
- **c)** Say which of the two counts in part (a) the answer to part (b) depended on.

D5-08 · REWRITING AND ROUTING FOR A MACHINE

A chip has five qubits joined in a line, $Q_0 - Q_1 - Q_2 - Q_3 - Q_4$, and the circuit asks for a CNOT between Q_0 and Q_4.

- **a)** Give the number of two-qubit gates the compiler needs, leaving the qubits permuted at the end.
- **b)** Give it again if the original layout has to be restored.
- **c)** Give it again if the chip is a ring, with Q_4 also joined to Q_0.

D5-09 · REWRITING AND ROUTING FOR A MACHINE

A compiler returns two valid versions of one circuit. Version A has 40 two-qubit gates and depth 30; version B has 52 and depth 18. Every layer contains at least one two-qubit gate, a two-qubit gate takes $400\,\mathrm{ns}$ and fails with probability 0.01, and the qubits have $T_2 = 60\,\mu\mathrm{s}$.

- **a)** Give the duration of each version.
- **b)** Give the probability that no gate fails, for each.
- **c)** Combine the two effects with a factor e^{-T/T_2} and say which version to run.

D5–10 · TELEPORTATION: BRANCHES, CORRECTIONS AND RESOURCES

Alice teleports $|\psi\rangle = 0.6\,|0\rangle + 0.8\,|1\rangle$ and reads $m_1 = 1$, $m_0 = 0$.

a) Give the state Bob holds before any correction.

b) Give the correction he applies, and verify that it returns $|\psi\rangle$.

c) Give the fidelity Bob would have if he applied nothing.

D5–11 · TELEPORTATION: BRANCHES, CORRECTIONS AND RESOURCES

Alice teleports an arbitrary qubit $|\psi\rangle$ with Bloch vector $\mathbf{r}$.

a) Give the probability of each of the four branches.

b) Give the four Bloch vectors Bob's qubit could carry, and their average.

c) Say whether Bob can tell that Alice ran the protocol at all, before the bits arrive.

D5–12 · TELEPORTATION: BRANCHES, CORRECTIONS AND RESOURCES

A three-qubit state is to be teleported, and a colleague proposes saving bandwidth by sending only m_1 for each qubit.

a) Give the entangled pairs and classical bits the honest protocol needs.

b) With only m_1 sent, give the state Bob ends up with, as a channel applied to ρ.

c) Give the resulting average fidelity over inputs spread across the sphere, and say what it shows.

D5–13 · TELEPORTATION: BRANCHES, CORRECTIONS AND RESOURCES

A group reports an average teleportation fidelity of $F_{\text{avg}} = 0.78$ over inputs spread across the sphere, using pairs of singlet fraction f and the relation $F_{\text{avg}} = (2f+1)/3$.

a) Say whether the result beats the classical benchmark.

b) Give the singlet fraction the result implies.

c) Give the singlet fraction that would be needed for $F_{\text{avg}} = 0.9$, and name two things the report must state besides the fidelity.

D5-14 · GROVER: ANGLE, ITERATIONS AND PROBABILITY

Four qubits carry $N = 16$ candidates, and exactly one is marked.

a) Give the angle θ in degrees.

b) Give the exact optimum r_* and the whole number nearest it.

c) Give the success probability at that whole number and at the one below it.

D5-15 · GROVER: ANGLE, ITERATIONS AND PROBABILITY

Six qubits carry $N = 64$ candidates.

a) With $M = 4$ marked, give θ, the best whole r, and P at it.

b) With $M = 1$ marked, give the same three numbers.

c) Give the ratio of the two iteration counts, and say what it shows.

D5-16 · GROVER: ANGLE, ITERATIONS AND PROBABILITY

Ten qubits carry $N = 1024$ candidates, one of them marked. A run is scheduled for as many iterations as the coherence time allows.

a) Give the best whole number of iterations and the success probability there.

b) Give the success probability after 50 iterations.

c) Give it after 75, and say what the three answers together mean for the schedule.

D5-17 · GROVER: ANGLE, ITERATIONS AND PROBABILITY

Two qubits carry $N = 4$ candidates and the marked one is $x = 2$, that is $|10\rangle$. The register starts in the uniform superposition and the oracle target is prepared in $|-\rangle$.

a) Give the state of the register after one oracle call.

b) Give the probability of measuring the marked state at that point.

c) Give the best whole number of iterations and the success probability there.

D5–18 · ASSESSING A RESOURCE CLAIM

A report states: "our quantum algorithm searches a ten-million-row customer database quadratically faster than any classical method".

a) Name the five components a resource claim must state, and say which the sentence states.

b) Say what the input model has to be for the quadratic saving to hold, and what that costs here.

c) Say what the correct classical baseline for this task is.

D5–19 · ASSESSING A RESOURCE CLAIM

One quantum oracle call takes $20\,\mu\mathrm{s}$ on the machine available; the same predicate takes $5\,\mathrm{ns}$ to evaluate on a classical processor. Grover needs about $\frac{\pi}{4}\sqrt{N}$ calls and the classical search about $N/2$.

a) Give both total times at $N = 10^6$.

b) Give the problem size at which the two are equal.

c) Say what would move that size, in each direction.

D5–20 · A FULL-LENGTH QUESTION: ONE PROTOCOL, END TO END

A group reports finding a marked item among $N = 4096$ candidates on hardware. Their oracle compiles to 65 two-qubit gates and their diffusion operator to 62. Two-qubit gates fail with probability 0.01 and there is no error correction.

a) Give θ, the exact optimum r_*, and the best whole number of iterations.

b) Give the success probability at that count, in the ideal circuit model.

c) Give the total number of two-qubit gates the run needs, and the probability that none of them fails.

d) Say how many shots would be needed to distinguish an ideal run from a uniformly random one, and what the answer to part (c) does to that plan.

e) Say which of the five components of a resource claim this report has and has not established.

MODULE 6

Quantum Algorithms

20 questions on quantum algorithms. Write your reasoning in the space under each one.

D6-01 · KICKBACK AND INTERFERENCE

The Deutsch circuit is run with the constant oracle $f(x) = 1$. The query qubit starts in $|0\rangle$ and the target in $|1\rangle$, both are sent through a Hadamard, the oracle acts, and a Hadamard is applied to the query qubit alone.

- **a)** Give the state of the two qubits just after the oracle.
- **b)** Give the state of the query qubit after the last Hadamard.
- **c)** Give the reading and say what it proves and what it does not.

D6-02 · KICKBACK AND INTERFERENCE

The Deutsch–Jozsa circuit is run on $n = 3$ query qubits with the oracle $f(x_2x_1x_0) = x_0$.

- **a)** Say whether the oracle satisfies the promise, and why.
- **b)** Give the amplitude of $|000\rangle$ at the output, and the state the circuit actually produces.
- **c)** Give the number of queries an exact classical algorithm needs in the worst case.

D6-03 · KICKBACK AND INTERFERENCE

A single control qubit in $|+\rangle$ controls the gate $S = \mathrm{diag}(1, i)$, whose target is prepared in $|1\rangle$.

- **a)** Give the eigenphase φ that the control picks up.
- **b)** Give the state of the control qubit after the gate, and the probability of reading 0 after a Hadamard.
- **c)** Say why one control qubit cannot tell this phase from $\varphi = 3/4$, and what measurement would.

D6-04 · PROMISE PROBLEMS AND QUERY COUNTS

A promised function on $n = 8$ bits is either constant or balanced.

a) Give the exact classical worst-case query count and the quantum query count.

b) Give the number of distinct classical queries needed to be wrong with probability below 10^{-3}.

c) Say what happens to each of those three counts as n grows.

D6-05 · PROMISE PROBLEMS AND QUERY COUNTS

The Deutsch–Jozsa circuit is run on $n = 4$ query qubits with a function that returns 1 on exactly six of its sixteen inputs. The promise has been broken.

a) Give the amplitude of $|0000\rangle$ at the output.

b) Give the probability of reading 0000.

c) Say what a reading of 0000 and a reading of anything else each allow you to conclude.

D6-06 · PROMISE PROBLEMS AND QUERY COUNTS

On three query qubits the oracle computes $f(x_2x_1x_0) = x_0 \oplus x_2$ into the target, using two CNOT gates.

a) Say whether the promise holds.

b) Give the string the circuit prints, and the probability of printing it.

c) Give the depth of the oracle and say whether the two CNOTs can run together.

D6-07 · THE TRANSFORM: AMPLITUDES AND COST

The three-qubit Fourier transform, $Q = 8$, is applied to the basis state $|6\rangle$.

a) Give the modulus of every output amplitude.

b) Give the angle the phase advances by as k increases by one.

c) Give the eight measurement probabilities, and say what a single run reveals about the input.

D6-08 · THE TRANSFORM: AMPLITUDES AND COST

The exact Fourier transform circuit is to be built on $n = 6$ qubits.

a) Give the number of Hadamards, the number of controlled rotations and the number of swaps.

b) Give the smallest rotation angle in the circuit.

c) A classical fast Fourier transform on 64 stored numbers costs about $Q \log_2 Q$ operations. Give that count and say why the two numbers should not be compared.

D6-09 · THE TRANSFORM: AMPLITUDES AND COST

The state $\frac{1}{\sqrt{2}} (|0\rangle + |4\rangle)$ on three qubits is sent through the Fourier transform, $Q = 8$.

a) Give the eight output amplitudes.

b) Give the eight measurement probabilities.

c) Say what one run tells you about the input, and how many runs a full description of the output would take.

D6-10 · PHASE ESTIMATION: PRECISION AND COST

Phase estimation is run with $t = 4$ counting qubits on a unitary whose eigenphase is $\varphi = 5/16$.

a) Give the reading and its probability.

b) Give the four bits the counting register holds.

c) Give the number of applications of U the run needs.

D6-11 · PHASE ESTIMATION: PRECISION AND COST

Phase estimation is run with $t = 3$ counting qubits on a unitary whose eigenphase is $\varphi = 0.1$.

a) Give the two readings that bracket the true phase.

b) Give the probability of each of them.

c) Compare their total with the guaranteed floor, and give the error of the best estimate.

D6-12 · PHASE ESTIMATION: PRECISION AND COST

A phase is to be estimated to six correct binary digits, with a failure probability of at most 0.05.

a) Give the number of counting qubits.

b) Give the number of applications of U that needs.

c) Say what changes if the failure probability is tightened to 0.005, and what changes if a seventh digit is wanted instead.

D6-13 · PHASE ESTIMATION: PRECISION AND COST

Phase estimation is run with $t = 12$ counting qubits, on a unitary whose own circuit is 800 gates.

a) Give the number of applications of U and the gate count they come to.

b) Give the gate count of the inverse Fourier transform.

c) Say what fraction of the circuit the transform is, and what a rewrite that halved it would buy.

D6-14 · ORDER FINDING AND ITS CLASSICAL STEPS

Take $N = 33$ and $a = 5$.

a) Give the order of 5 modulo 33.

b) Check the two conditions the factoring step needs.

c) Give the two factors, and verify them.

D6-15 · ORDER FINDING AND ITS CLASSICAL STEPS

Order finding is run on $N = 21$ with $a = 2$. Take the idealised case in which every run samples one value of s uniformly and the continued fractions return the reduced form of s/r.

a) Give the order, the eigenphases, and the probability of each.

b) For each measured s, give the candidate order the continued fractions return and say whether it passes the check.

c) Give the probability that one run succeeds.

D6-16 · ORDER FINDING AND ITS CLASSICAL STEPS

An order-finding run on $N = 15$ with $a = 2$ uses $t = 8$ counting qubits and returns the reading $y = 192$.

a) Give the fraction y/Q and its continued-fraction convergents.

b) Give the candidate order and confirm it.

c) Give the two factors of fifteen.

D6-17 · ORDER FINDING AND ITS CLASSICAL STEPS

An order-finding run on $N = 21$ with $a = 2$ uses $t = 9$ counting qubits and returns $y = 427$.

a) Give the convergents of y/Q.

b) Give the candidate order and confirm it.

c) Say why $t = 9$ was chosen rather than $t = 5$.

D6-18 · ASSESSING AN ALGORITHMIC CLAIM

A summary states: "the Deutsch–Jozsa algorithm gives an exponential speedup over classical computers."

a) Name the five components a resource claim has to state, and say which of them this sentence states.

b) Rewrite the sentence so that it is true.

c) Give the largest honest ratio for $n = 20$, and say against which classical algorithm.

D6-19 · ASSESSING AN ALGORITHMIC CLAIM

A resource estimate for factoring a 1024-bit modulus is to be sketched. Take a reversible modular multiplier to cost about L^2 gates, and a counting register of $t = 2L$ qubits.

a) Give the number of controlled modular multiplications and the gates they come to.

b) Give the gate count of the inverse Fourier transform on that register.

c) Say which of the two an estimate is really an estimate of, and what the hardware model adds.

D6-20 · A FULL-LENGTH QUESTION: ONE ALGORITHM, END TO END

Factor $N = 21$ by order finding, with the base $a = 2$. Take the counting register to be as short as the uniqueness condition allows, and assume every reading is exact.

a) Give the counting register size and the number of applications of the modular multiplier.

b) Give the order, the eigenphases, and the probability that one run yields a usable reading.

c) Take the reading $y = 427$ and recover the order from it.

d) Give the two factors, and verify them.

e) State the claim this run supports and the claim it does not, using the five components.